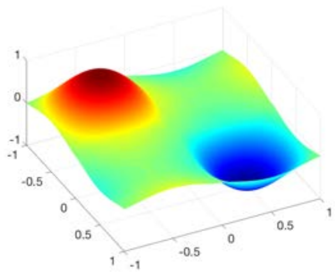
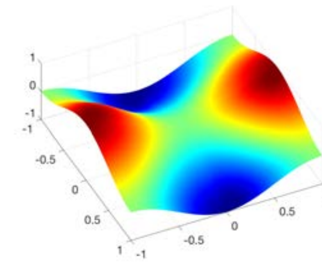
Part II

Dirac, Dirac-like, and almost-Dirac points

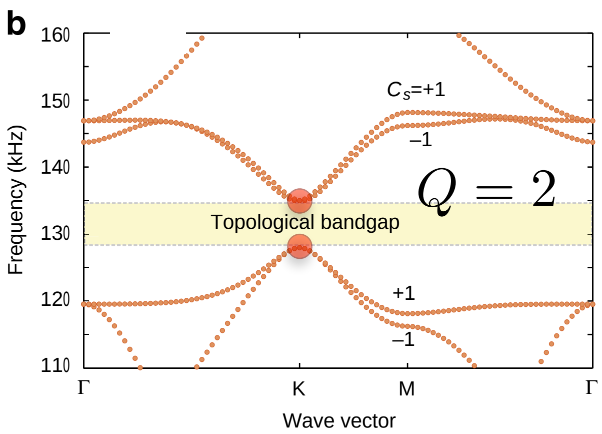
G/Meng/Oudghiri-Idrissi

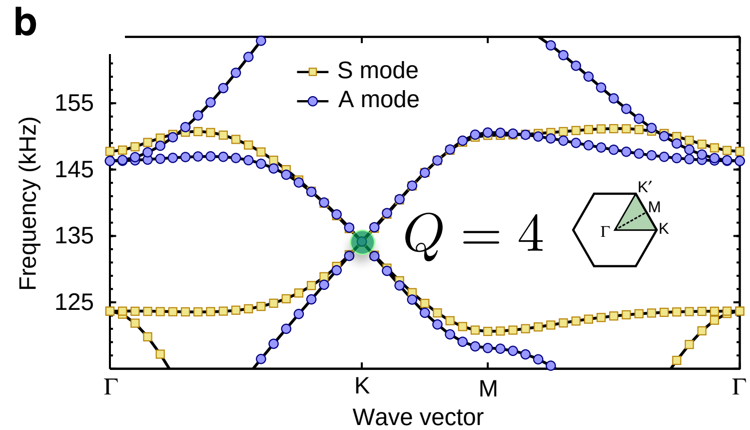
Repeated eigenvalues





Q projections (phonons)



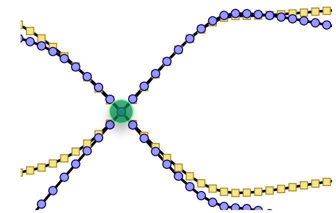


Mousavi et al. (2015)

O(ϵ^{-2}): $$-\tilde{\lambda}_n^{\boldsymbol{a}} \rho \tilde{w}_0 - \nabla\cdot\left(G\nabla\tilde{w}_0\right) \;=\; 0 \quad \text{in} \quad Y_{\boldsymbol{a}}$$ ⇨ $$\tilde{w}_0(\boldsymbol{x}) \;=\; w_0\, \tilde{\varphi}_n^{\boldsymbol{a}}(\boldsymbol{x})$$

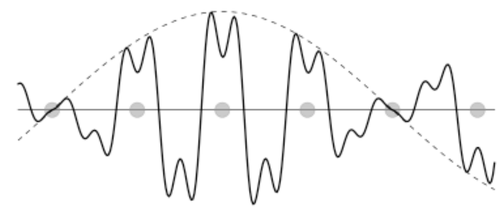
$$\sum_q w_{0q}\, \tilde{\varphi}_{nq}^{\boldsymbol{a}}(\boldsymbol{x})$$

Energy



$$\left(\rho\tilde{\varphi}_{np}^{\boldsymbol{a}}, \tilde{\varphi}_{nq}^{\boldsymbol{a}}\right)_{Y_{\boldsymbol{a}}} = 0 \quad \text{for } p \neq q$$

Bloch wave



$$u(\boldsymbol{x}) \;=\; \tilde{u}(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \tilde{u} : Y_{\boldsymbol{a}}\text{-periodic}$$

Mean energy density

Willis (2016) *JMPS*, **97**

$$\bar{E} \;=\; \bar{E}_p + \bar{E}_k \;=\; \tfrac{1}{2}\omega^2(\rho\tilde{u}, \tilde{u})_{Y_{\boldsymbol{a}}}$$

Repeated eigenvalue

$$\tilde{w}_0(\boldsymbol{x}) \;=\; \sum_q w_{0q}\tilde{\varphi}_{nq}^{\boldsymbol{a}}(\boldsymbol{x})$$

Quanta of energy

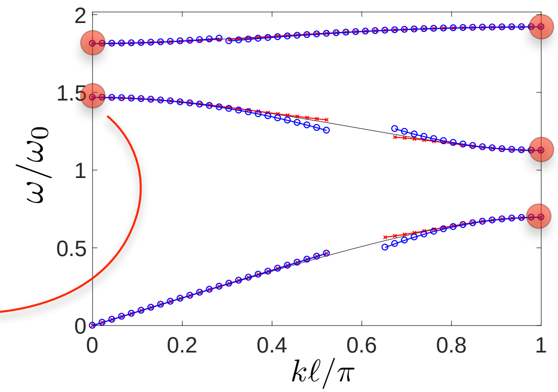
$$\rho_p^{(0)} \;=\; \langle \rho\tilde{\varphi}_{np}^{\boldsymbol{a}} \rangle_{\boldsymbol{a}}^{p\varphi}$$

$$\bar{E}_0 \;=\; \tfrac{1}{2}\omega^2(\rho\tilde{u}_0, \tilde{u}_0) \;=\; \tfrac{1}{2}\omega^2\tilde{f}^2\sum_q \rho_q^{(0)} w_{0q}^2$$

Scaling

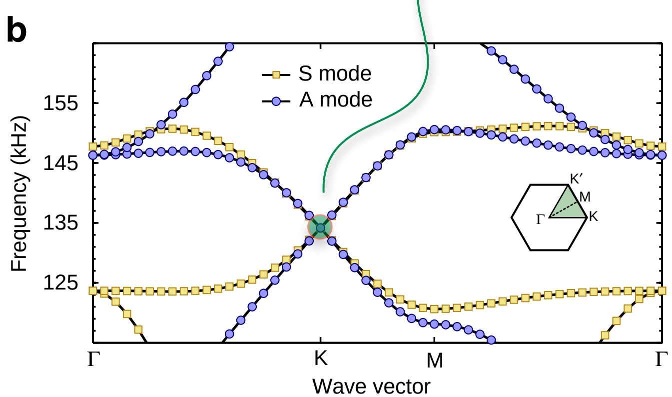
Isolated eigenvalues

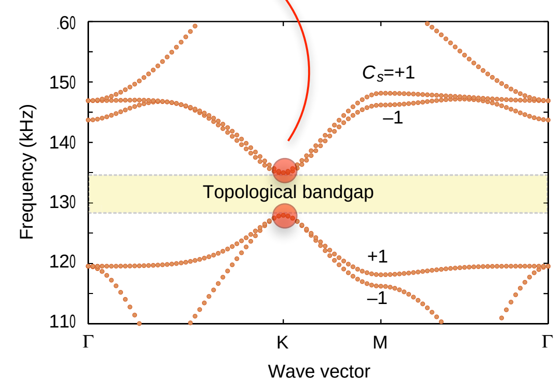
$$\boldsymbol{k} = \boldsymbol{k}^{\boldsymbol{a}} + \epsilon \hat{\boldsymbol{k}}, \qquad \omega^2 = \tilde{\lambda}_n^{\boldsymbol{a}} + \epsilon^2 \sigma \hat{\omega}^2$$



Repeated eigenvalues

$$\boldsymbol{k} = \boldsymbol{k}^{\boldsymbol{a}} + \epsilon \hat{\boldsymbol{k}}, \qquad \omega^2 = \tilde{\lambda}_n^{\boldsymbol{a}} + \epsilon \sigma \breve{\omega}^2 + \epsilon^2 \sigma \hat{\omega}^2$$





Cone or no cone?

Ansatz

O(ϵ^{-2}): $$-\tilde{\lambda}_n^{\boldsymbol{a}} \rho \tilde{w}_0 - \nabla\cdot\left(G\nabla\tilde{w}_0\right) = 0 \quad \text{in} \quad Y_{\boldsymbol{a}} \quad \Rightarrow \quad \tilde{w}_0(\boldsymbol{x}) = w_0 \tilde{\varphi}_n^{\boldsymbol{a}}(\boldsymbol{x})$$

$$\sum_q w_{0q} \tilde{\varphi}_{nq}^{\boldsymbol{a}}(\boldsymbol{x})$$

O(ϵ^{-1}): ➡ Identity

O(ϵ^{0}): ➡ Effective equation

$$\sum_q \boldsymbol{\theta}_{pq}^{(0)} \cdot (i\hat{\boldsymbol{k}})\, w_{0q} + \sigma\breve{\omega}^2 \rho_p^{(0)} w_{0p} = 0, \qquad p = \overline{1,Q}$$

$$\boldsymbol{\theta}_{pq}^{(0)} = \left\langle G\nabla\tilde{\varphi}_{nq}^{\boldsymbol{a}}\right\rangle_{\boldsymbol{a}}^{p\varphi} - \left\langle G\nabla\tilde{\varphi}_{np}^{\boldsymbol{a}}\right\rangle_{\boldsymbol{a}}^{q\varphi}$$

Generalized EVP

imaginary antisymmetric

$$\sum_q A_{pq} w_{0q} - \tau \sum_q D_{pq} w_{0q} = 0, \qquad p = \overline{1,Q}$$

Effective models

$\tau = -\sigma\breve{\omega}^2$

Generalized EVP

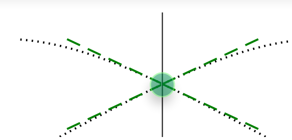
$$\sum_q A_{pq} w_{0q} - \tau \sum_q D_{pq} w_{0q} = 0, \qquad p = \overline{1, Q}$$

$$A_{pq} = \boldsymbol{\theta}^{(0)}_{pq} \cdot i\hat{\boldsymbol{k}}, \qquad \boldsymbol{\theta}^{(0)}_{pq} = \langle G \nabla \tilde{\varphi}^{\boldsymbol{a}}_{nq} \rangle^{p\varphi}_{\boldsymbol{a}} - \langle G \nabla \tilde{\varphi}^{\boldsymbol{a}}_{np} \rangle^{q\varphi}_{\boldsymbol{a}}$$

Full-rank A_{pq}: $w_{0q}=0$ & $\tau \neq 0$, system of *1st-order* PDE's for w_{1q}

$$-\sum_q A_{pq} w_{1q} - \sigma\breve{\omega}^2 \sum_q D_{pq} w_{1q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}} \cdot \boldsymbol{x}} \rangle^{p\varphi}_{\boldsymbol{a}}, \qquad p = \overline{1, Q}$$

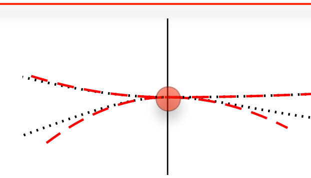
Dirac



Trivial A_{pq}: $w_{0q} \neq 0$ & $\tau = 0$, system of *2nd-order* PDE's for w_{0q}

$$-\sum_q B_{pq} w_{0q} - \sigma\hat{\omega}^2 \sum_q D_{pq} w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}} \cdot \boldsymbol{x}} \rangle^{p\varphi}_{\boldsymbol{a}}, \qquad p = \overline{1, Q}$$

Wave

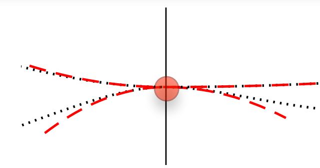


$$B_{pq} = \boldsymbol{\mu}^{(0)}_{pq} : (i\hat{\boldsymbol{k}})^2, \qquad \boldsymbol{\mu}^{(0)}_{pq} = \langle G\{\nabla \boldsymbol{\chi}^{(1)}_q + \boldsymbol{I}\tilde{\varphi}^{\boldsymbol{a}}_{nq}\} \rangle^{p\varphi}_{\boldsymbol{a}} - (G\{\boldsymbol{\chi}^{(1)}_q \otimes \nabla \tilde{\varphi}^{\boldsymbol{a}}_{np}\}, 1)_{\overline{Y_{\boldsymbol{a}}}}$$

Caution

Trivial A_{pq}: $w_{0q} \neq 0$ & $\tau = 0$, system of *2nd-order* PDE's for w_{0q}

$$-\sum_q B_{pq} w_{0q} - \sigma \hat{\omega}^2 \sum_q D_{pq} w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}} \rangle^{p\varphi}_{\boldsymbol{a}}, \qquad p = \overline{1, Q}$$



$$B_{pq} = \boldsymbol{\mu}^{(0)}_{pq} : (i\hat{\boldsymbol{k}})^2, \qquad \boldsymbol{\mu}^{(0)}_{pq} = \langle G\{\nabla \boldsymbol{\chi}^{(1)}_q + \boldsymbol{I}\tilde{\varphi}^{\boldsymbol{a}}_{nq}\} \rangle^{p\varphi}_{\boldsymbol{a}} - (G\{\boldsymbol{\chi}^{(1)}_q \otimes \nabla \tilde{\varphi}^{\boldsymbol{a}}_{np}\}, 1)_{\overline{Y}_{\boldsymbol{a}}}$$

$$\tilde{w}_0(\boldsymbol{x}) = \sum_q w_{0q} \tilde{\varphi}^{\boldsymbol{a}}_{nq}(\boldsymbol{x})$$

$$\tilde{w}_1(\boldsymbol{x}) = \sum_q w_{0q} \boldsymbol{\chi}^{(1)}_q(\boldsymbol{x}) \cdot i\hat{\boldsymbol{k}} + \sum_q w_{1q} \tilde{\varphi}^{\boldsymbol{a}}_{nq}(\boldsymbol{x})$$

$$\boldsymbol{\chi}^{(1)}_q \in (H^{1\boldsymbol{a}}_{p0}(Y_{\boldsymbol{a}}))^d$$

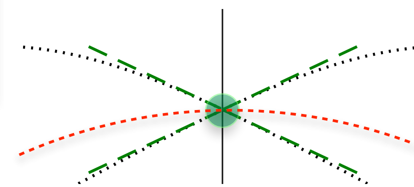
$$\tilde{\lambda}^{\boldsymbol{a}}_n \rho \boldsymbol{\chi}^{(1)}_q + \nabla \cdot \big(G(\nabla \boldsymbol{\chi}^{(1)}_q + \boldsymbol{I}\tilde{\varphi}^{\boldsymbol{a}}_{nq})\big) + G\nabla \tilde{\varphi}^{\boldsymbol{a}}_{nq} - \sum_r \frac{1}{\rho^{(0)}_r} \boldsymbol{\theta}^{(0)}_{rq} \rho \tilde{\varphi}^{\boldsymbol{a}}_{nr} = 0 \quad \text{in } Y_{\boldsymbol{a}}, \quad q = \overline{1, Q}$$

$$\boldsymbol{\nu} \cdot G(\nabla \boldsymbol{\chi}^{(1)}_q + \boldsymbol{I}\tilde{\varphi}^{\boldsymbol{a}}_{nq})|_{x_j = 0} = -\boldsymbol{\nu} \cdot G(\nabla \boldsymbol{\chi}^{(1)}_q + \boldsymbol{I}\tilde{\varphi}^{\boldsymbol{a}}_{nq})|_{x_j = (1+a_j)\ell_j}.$$

Mixed degeneracy

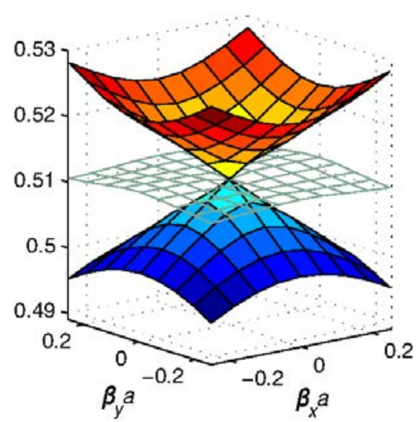
Partial rank A_{pq} (Q-1):

$$-\left(\boldsymbol{\mu}_{11}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \sigma\hat{\omega}^2\rho_1^{(0)}\right)\mathrm{w}_0 = \cdots$$ 1 parabola

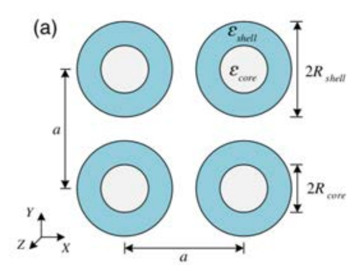


$$-\sum_{q=2}^{Q} A_{pq}w_{1q} - \sigma\breve{\omega}^2\sum_{q=2}^{Q} D_{pq}w_{1q} = \cdots$$ Q-1 cones

Ashraf & Faryad (2015)
J. Nanophotonics, **9**

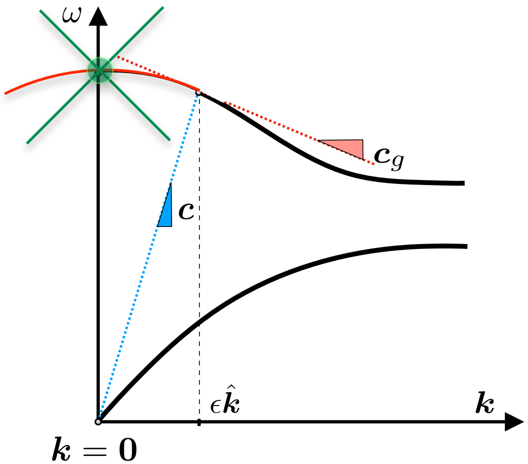


"Dirac-like cones"



Zero-index metamaterials

EM waves:
$\varepsilon=0$ and/or $\mu=0$

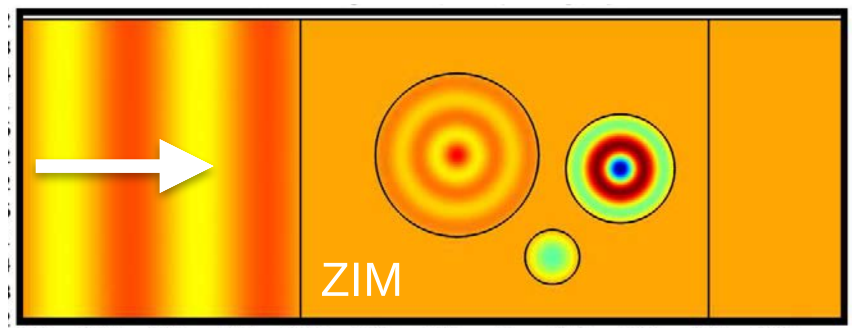


Phase velocity $c \to \infty$ (zero refraction index)

Applications to cloaking and shielding

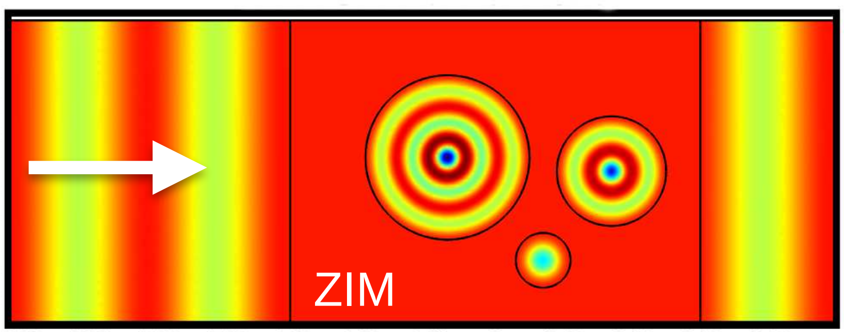
Ziolkowski (2004) *Phys. Rev. E,* **70**
Nguyen, Chen & Halterman (2010), *Phys. Rev. Lett.,* **105**
Chan, Hang & Huang (2012), *Adv. Optoelectronics*
Ashraf & Faryad (2015) *J. Nanophotonics*, **9**

Shielding

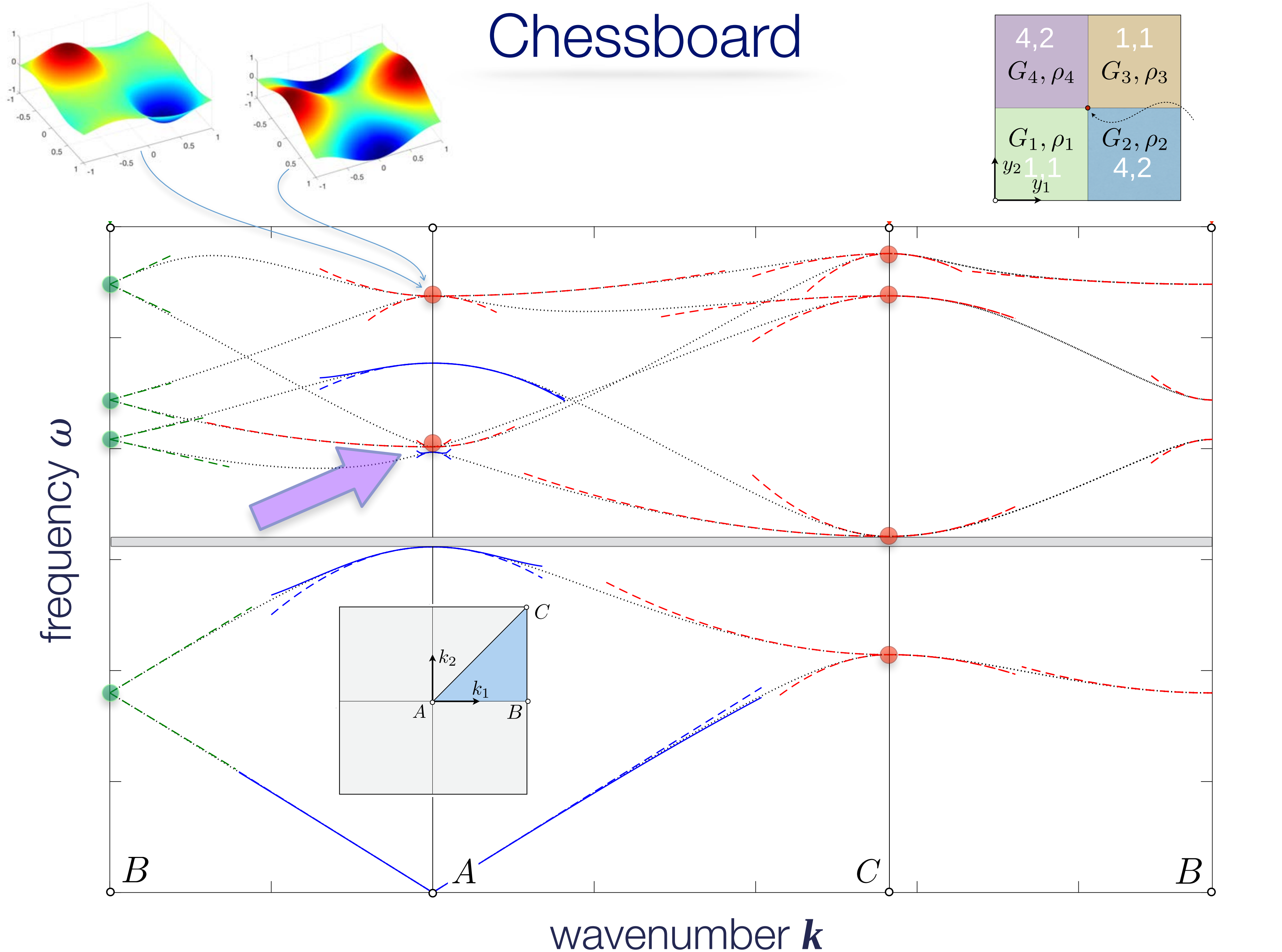


Nguyen, Chen & Halterman (2010)

Cloaking



Chessboard

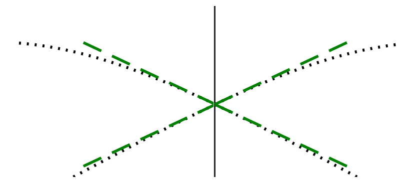


4,2
G_4, ρ_4
1,1
G_3, ρ_3
G_1, ρ_1
G_2, ρ_2
y_2
1,1
y_1
4,2
frequency ω
wavenumber $\boldsymbol{k}$
B
A
C
B
k_2
k_1
A
B
C

Clusters of nearby eigenvalues

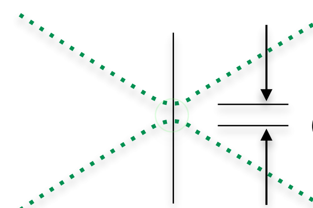
Repeated eigenvalue

$$-\sum_q A_{pq} w_{1q} - \sigma\breve{\omega}^2 \sum_q D_{pq} w_{1q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}}$$



Nearby eigenvalues?

$$\tilde{\lambda}^{\boldsymbol{a}}_{j(q)} = \tilde{\lambda}^{\boldsymbol{a}}_n - \epsilon\gamma_q, \quad q = \overline{1,Q}$$



$$A_{pq} \rightarrow A^{\gamma}_{pq} = \boldsymbol{\theta}^{(0)}_{pq}\cdot i\hat{\boldsymbol{k}} + \gamma_q D_{pq}$$

$$\begin{bmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & A_{12} \\ -A_{12} & \gamma_2\rho_2^{(0)} \end{bmatrix}$$

"blunted cones"

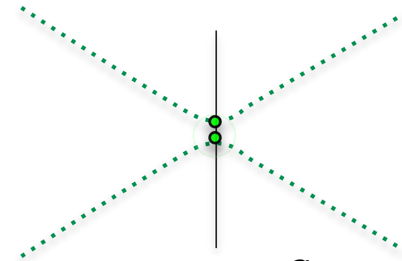
Cluster

Nearby eigenvalues

$$-\nabla\cdot\left(G\nabla\tilde{\varphi}^{\boldsymbol{a}}_{nq}\right) = \tilde{\lambda}^{\boldsymbol{a}}_{n}\,\rho\,\tilde{\varphi}^{\boldsymbol{a}}_{nq} + \epsilon\gamma_q\,\rho\,\tilde{\varphi}^{\boldsymbol{a}}_{nq} \quad \text{in } Y_{\boldsymbol{a}},$$

$$a_j\tilde{\varphi}^{\boldsymbol{a}}_{nq}|_{x_j=0} = -a_j\tilde{\varphi}^{\boldsymbol{a}}_{nq}|_{x_j=\ell_j},$$

$$\boldsymbol{\nu}\cdot G\nabla\tilde{\varphi}^{\boldsymbol{a}}_{nq}|_{x_j=0} = -\boldsymbol{\nu}\cdot G\nabla\tilde{\varphi}^{\boldsymbol{a}}_{nq}|_{x_j=(1+a_j)\ell_j}$$



$$\boldsymbol{k} = \boldsymbol{k}^{\boldsymbol{a}}$$

Governing equation

O(ϵ^{-2}): $$-\tilde{\lambda}^{\boldsymbol{a}}_{n}\,\rho\,\tilde{w}_0 - \nabla\cdot\left(G\nabla\tilde{w}_0\right) = 0 \quad \text{in } Y_{\boldsymbol{a}}$$

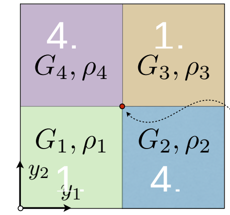
Leading-order solution

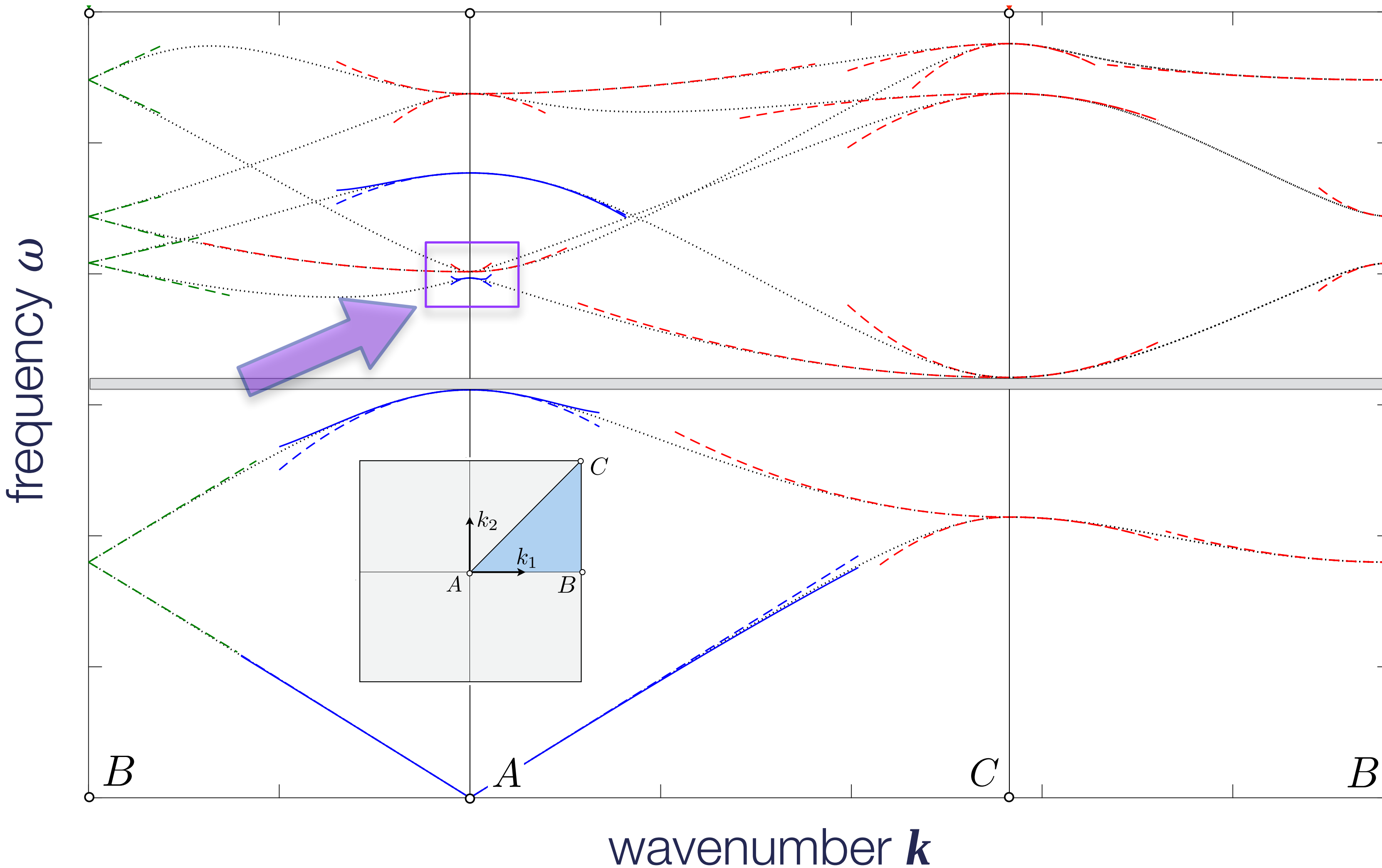
$$\tilde{w}_0(\boldsymbol{x}) = \sum_q w_{0q}\,\tilde{\varphi}^{\boldsymbol{a}}_{nq}(\boldsymbol{x})$$

Residual

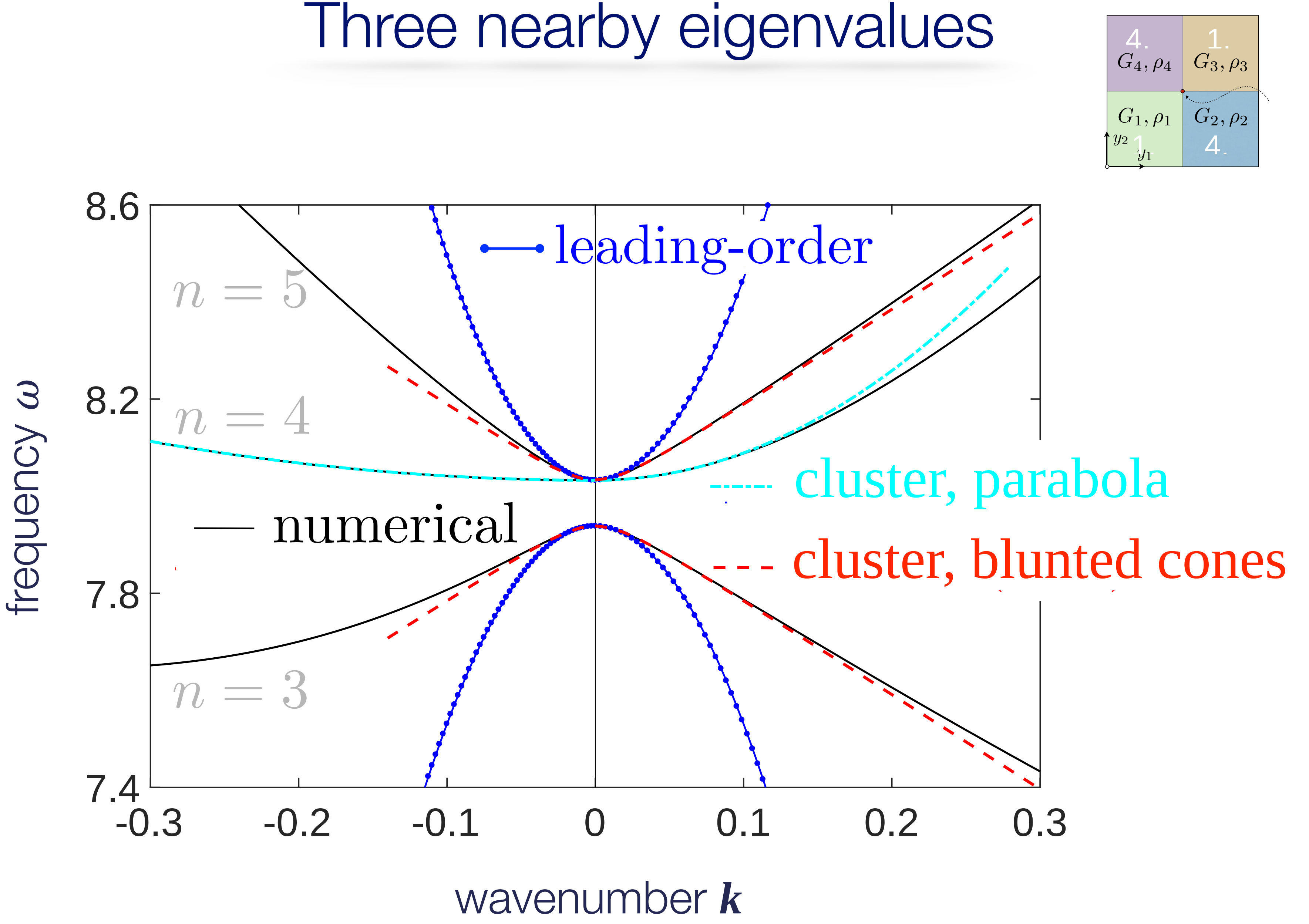
$$\epsilon\,\rho(\boldsymbol{x})\sum_q \gamma_q\,w_{0q}\,\tilde{\varphi}^{\boldsymbol{a}}_{nq}(\boldsymbol{x}) \quad \longrightarrow \quad A^{\gamma}_{pq} = \boldsymbol{\theta}^{(0)}_{pq}\cdot i\hat{\boldsymbol{k}} + \gamma_q\,D_{pq}$$

Redux

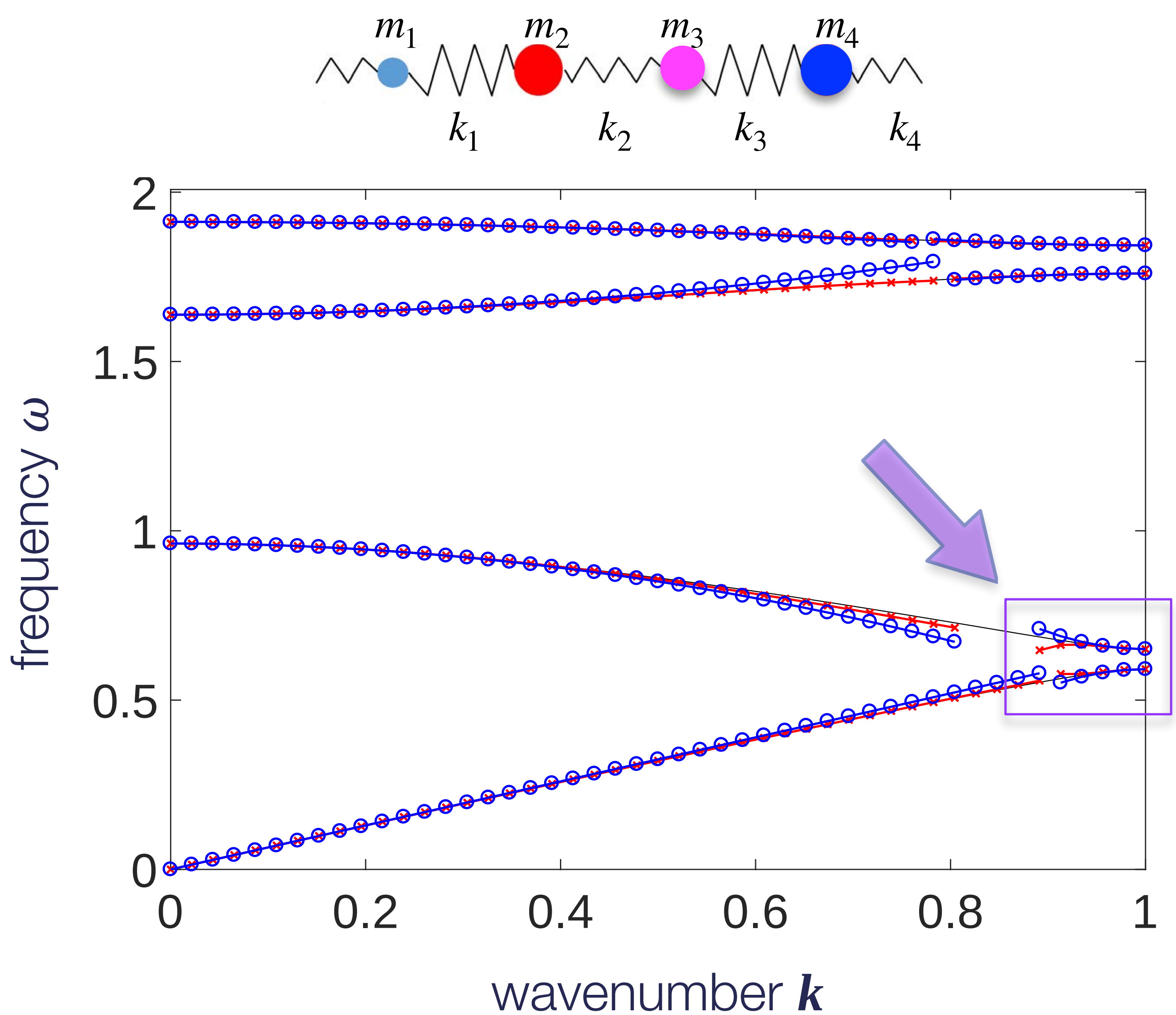

4.
G_4, ρ_4
1.
G_3, ρ_3
G_1, ρ_1
G_2, ρ_2
y_2
y_1
1.
4.


frequency ω
wavenumber $\boldsymbol{k}$
B
A
C
B
C
k_2
k_1
A
B

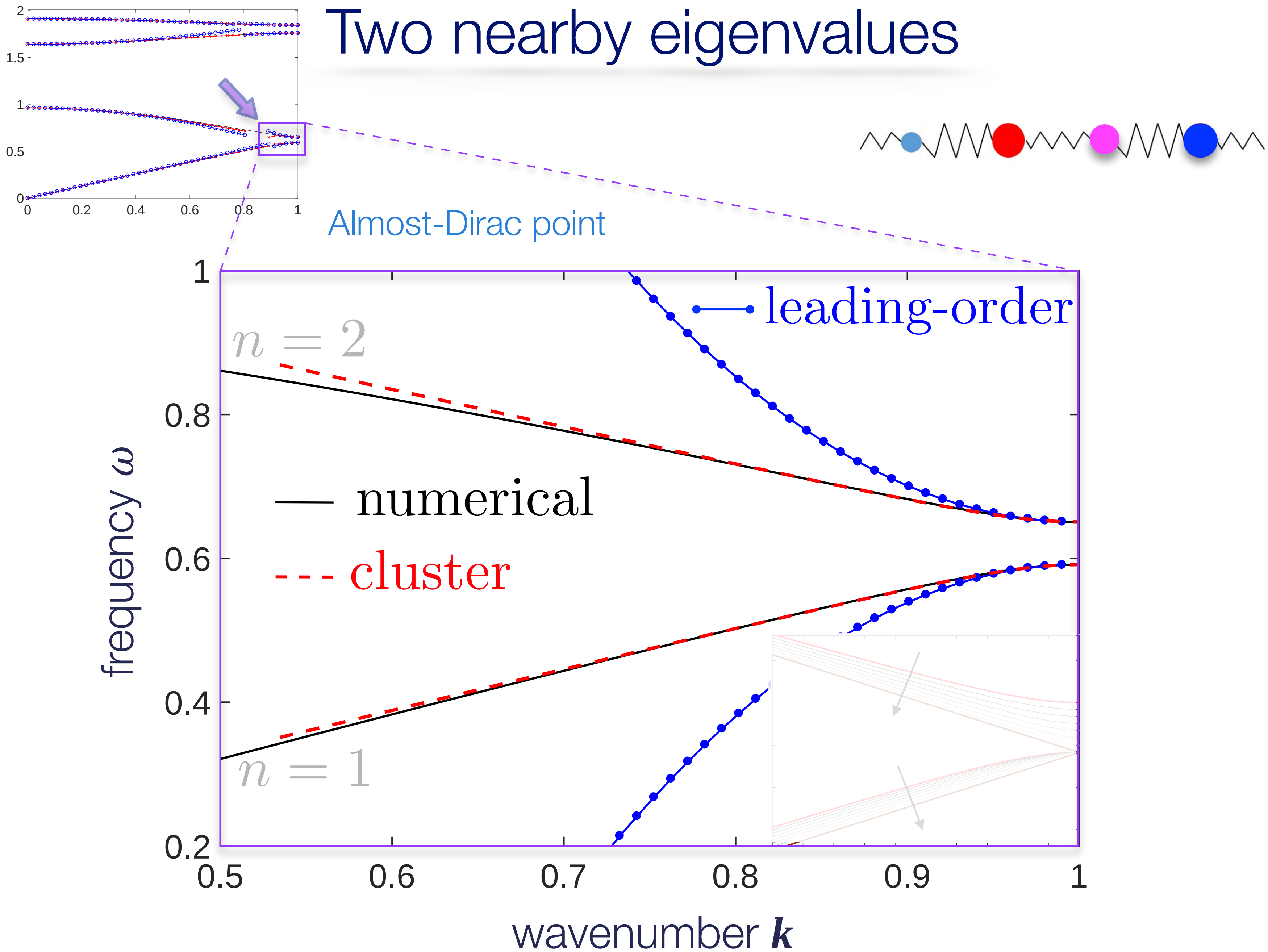
Three nearby eigenvalues



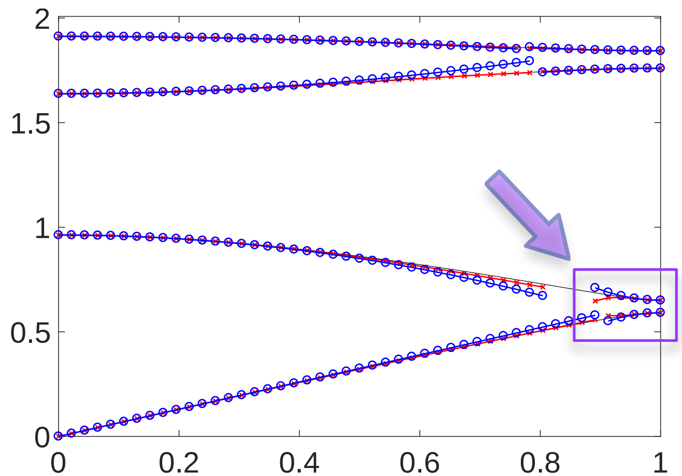
Tetratomic chain



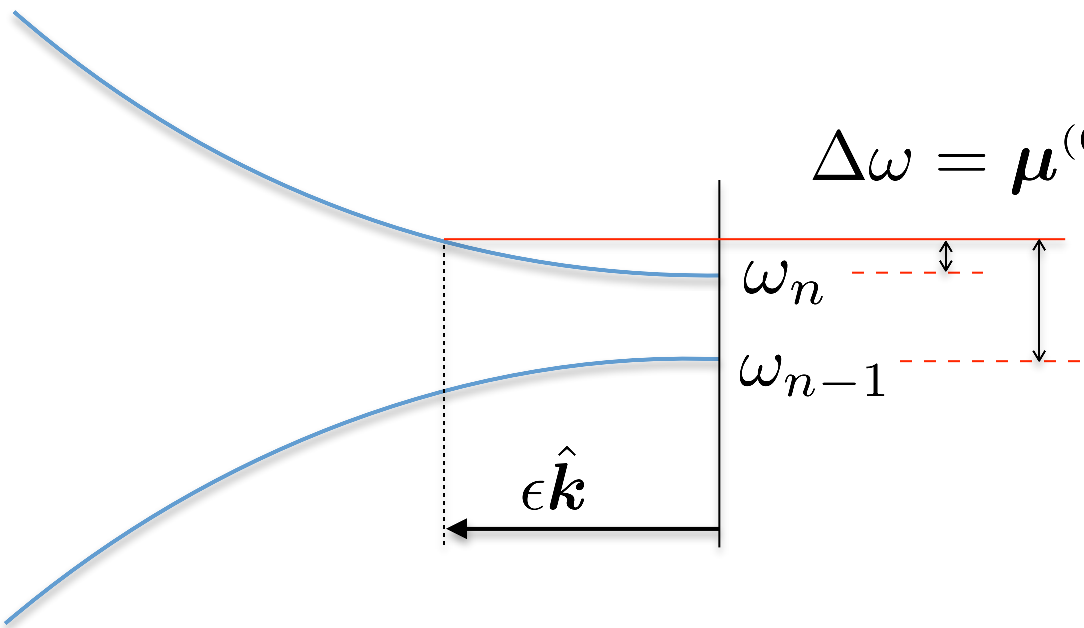
Two nearby eigenvalues



Role of the curvature

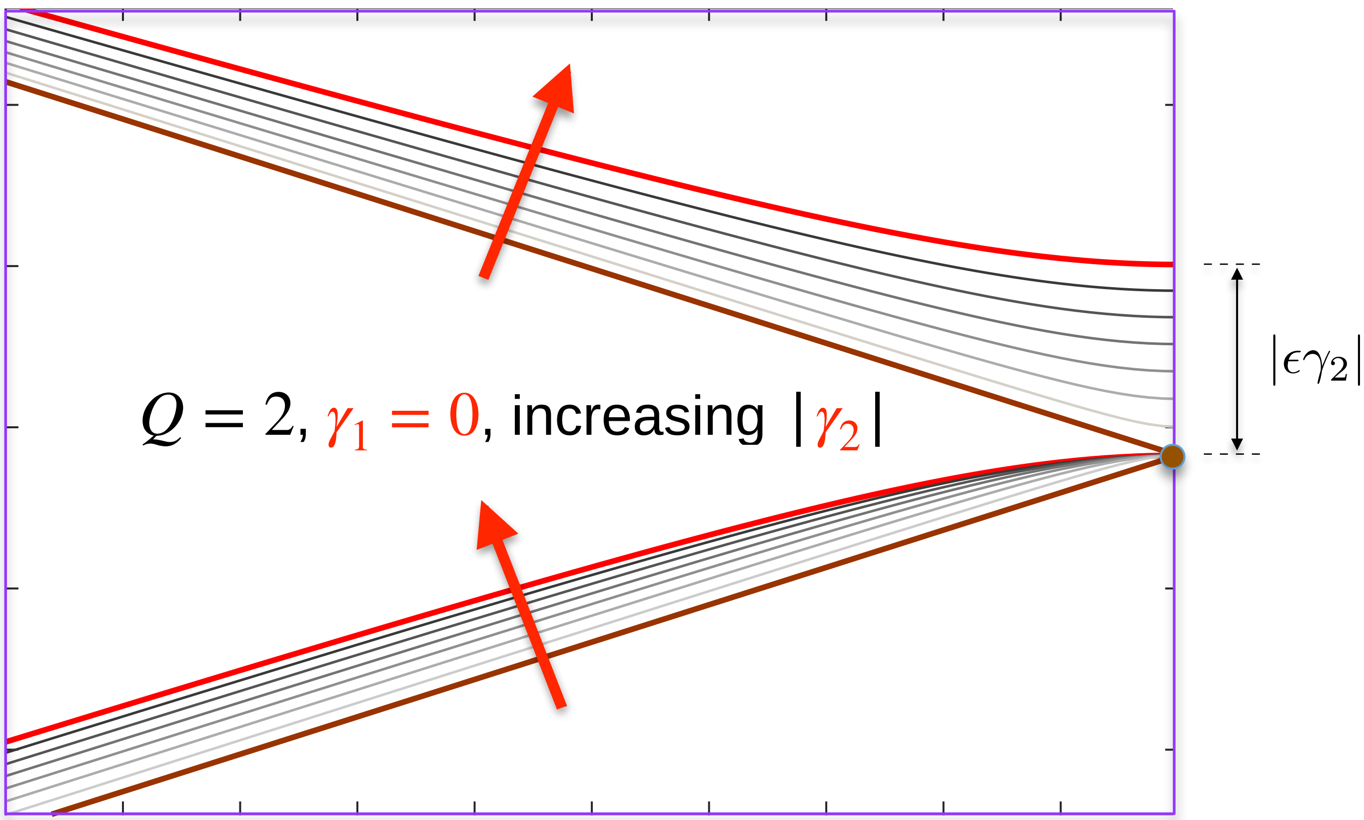


$$\Delta\omega = \boldsymbol{\mu}^{(0)}\colon(i\hat{\boldsymbol{k}})^2 \ll \omega - \omega_{n-1}$$



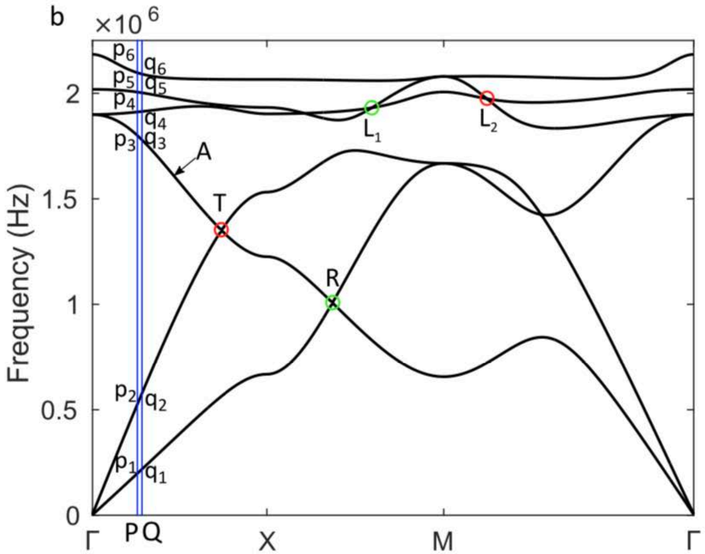
Dirac is (almost) everywhere

$$-\sum_q A_{pq} w_{1q} - \sigma \breve{\omega}^2 \sum_q D_{pq} w_{1q} = 0$$

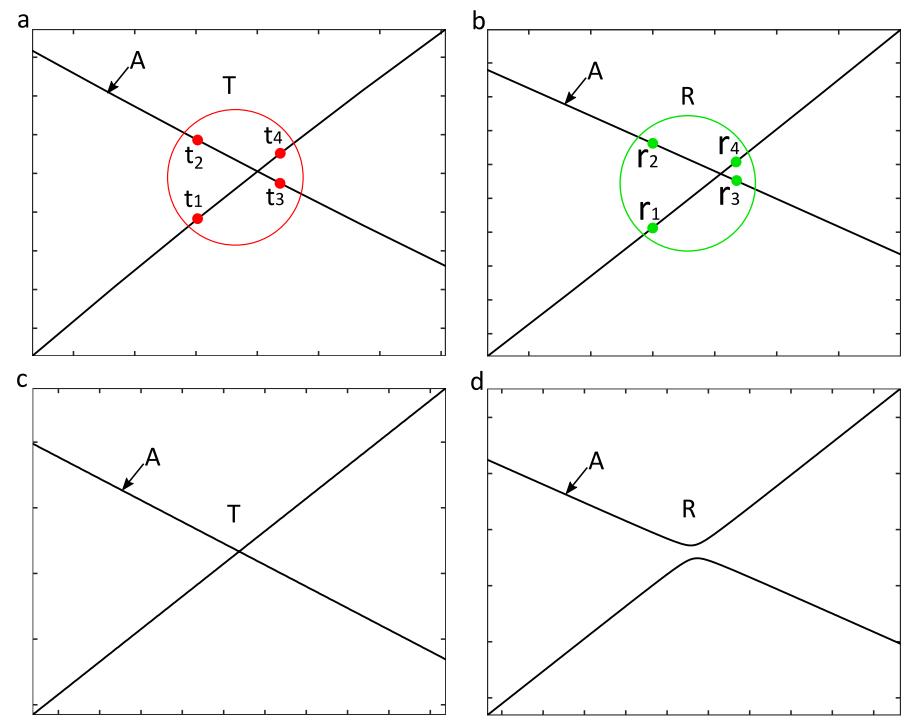


$$\begin{bmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & A_{12} \\ -A_{12} & \gamma_2 \rho_2^{(0)} \end{bmatrix}$$

Band sorting & level repulsion



Lu & Srivastava (2018) *JMPS*, **111**



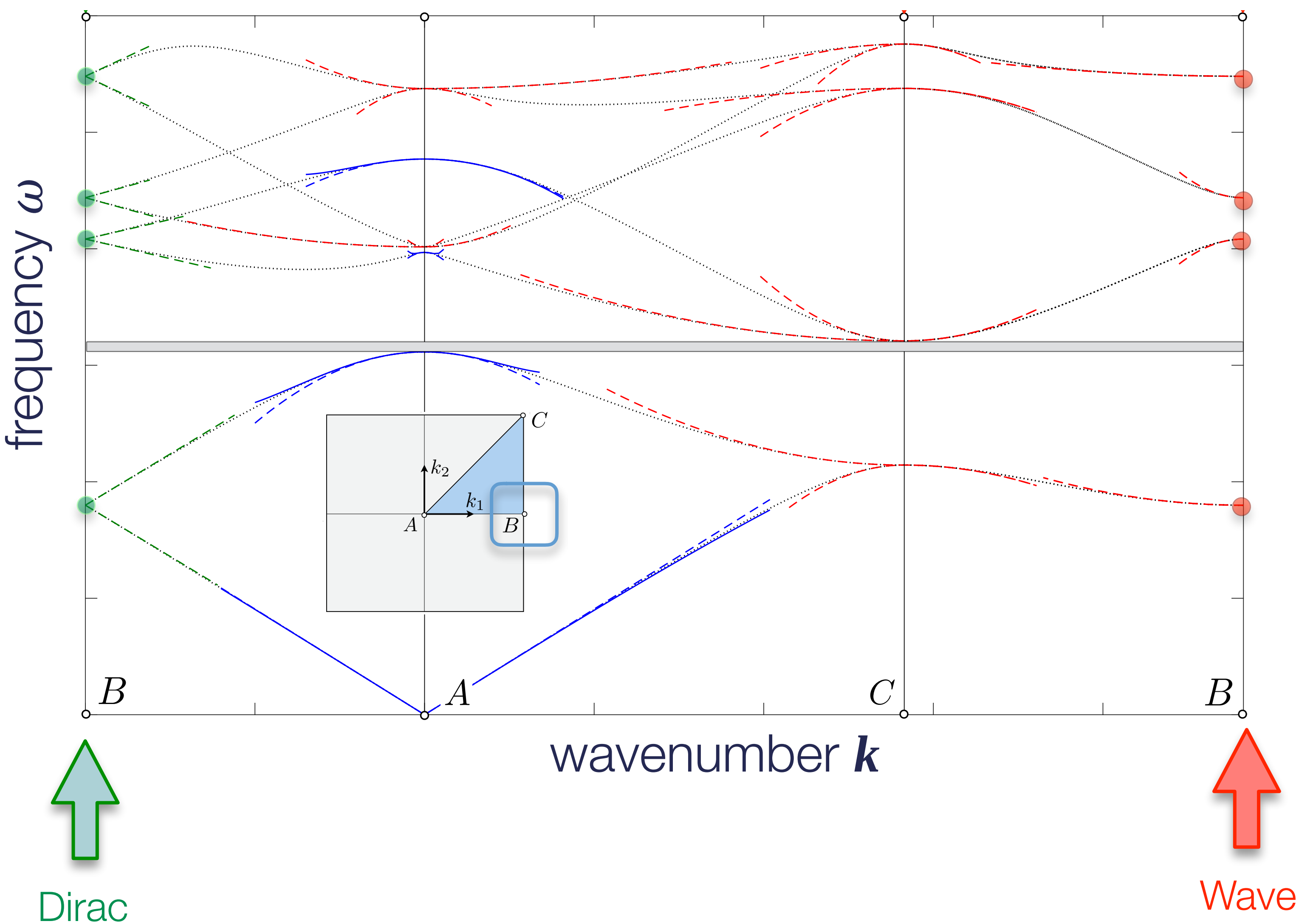
Macroscopic view

?

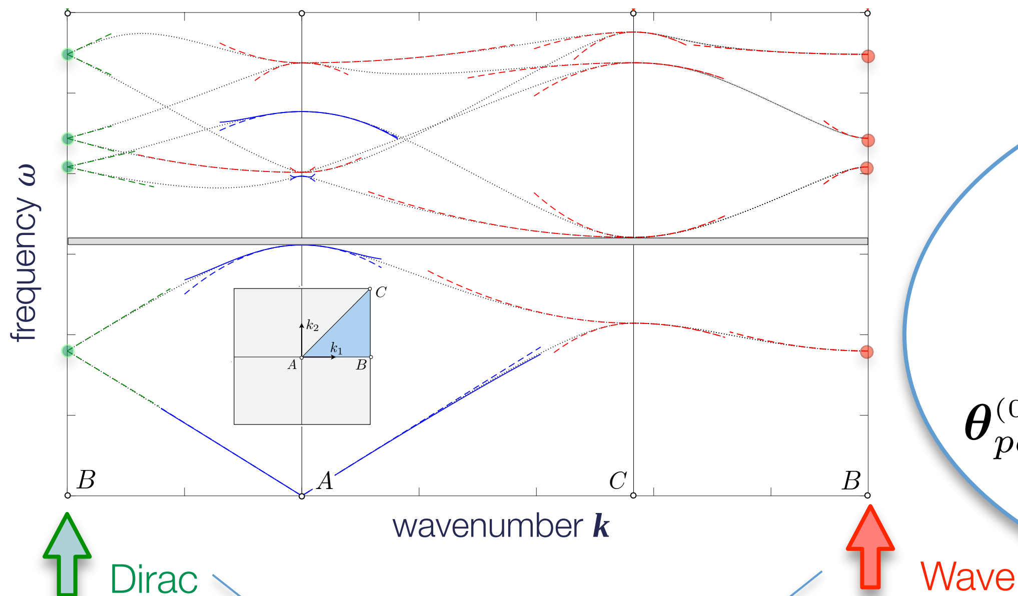
Zoom-in

Remarks

Direction-dependent asymptotic behavior



Remarks



Uniform asymptotics?

$$\begin{bmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{bmatrix}$$

$$A_{pq} = \boldsymbol{\theta}_{pq}^{(0)} \cdot i\hat{\boldsymbol{k}}$$

$$\boldsymbol{\theta}_{pq}^{(0)} = \langle G\nabla\tilde{\varphi}_{nq}^{\boldsymbol{a}} \rangle_{\boldsymbol{a}}^{p\varphi} - \langle G\nabla\tilde{\varphi}_{np}^{\boldsymbol{a}} \rangle_{\boldsymbol{a}}^{q\varphi}$$

Lemma 5.11. *A sufficient condition for the occurrence of a Dirac point* [15], *i.e. conical contact between dispersion surfaces, at* $(\boldsymbol{k}^{\boldsymbol{a}}, \omega_n^{\boldsymbol{a}})$ *is given by*

$$\operatorname{rank}(A_{pq}) = Q \quad \forall \frac{\hat{\boldsymbol{k}}}{\|\hat{\boldsymbol{k}}\|} \in \{\boldsymbol{\kappa} \in \mathbb{R}^d \colon \|\boldsymbol{\kappa}\| = 1\}, \quad Q > 1.$$

As a result, 'simple' $(Q = 2)$ Dirac points of the scalar wave equation are not possible.

Proof. When $Q = 2$, the eigenvalues of A_{pq} are $\pm\boldsymbol{\theta}_{12}^{(0)} \cdot \hat{\boldsymbol{k}}$, which vanish for $\hat{\boldsymbol{k}}/\|\hat{\boldsymbol{k}}\| \perp \boldsymbol{\theta}_{12}^{(0)}$. ∎