Modeling the response of structured gyro-elastic systems

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Wave Propagation in Complex and Microstructured Media,
Cargése

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Background: Steady state analysis of chiral periodic structures


- wave polarisation
- dynamic anisotropy
Background: Applications and experiments


Recent advances: Gaussian beams, localised waveforms

The DASER
Dynamic Amplification by Spinners in Elastic Reticulated systems

Recent advances: Uni-directional interfacial waveforms

Assuming

(i) small nutation $\theta$ and a constant spin rate $\dot{\psi}$.

(ii) Moments applied to the gyroscope about the $x$ and $y$ axes are zero.

$\Rightarrow \dot{\phi}$ is constant and $\dot{\phi} \propto \omega$

In the time-harmonic regime, the equation of motion can be written as

$$-m\omega^2 u = -Ku + i\beta \omega^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u$$

$\beta = I_z/h^2$ – Spinner constant

$I_z$ – Moment of Inertia of gyro about $z$

$h$ – "height" mass from base of gyroscope

(1)

Gyrobeams
These are flexural elements with “additional stored angular momentum”.

Transient analysis of a gyro-elastic lattice

Gyro-elastic lattice

Model of a mass connected to several rods and a gyroscopic spinner

We aim to derive a linearised transient model describing the motion of the mass in this system. This requires:

i) Linear momentum balance for the mass.

ii) Angular momentum balance of the gyroscope.

iii) Assumptions governing the motion of the system.
Angular momentum balance for the gyroscope

Angular momentum balance:

\[ M_e = \frac{d}{dt}(I_g \omega_g) \]

External moments:

\[ M_e = l \times F \]

(\( l \) is the vector representing the arm of \( F' \))

Moment of inertia tensor:

\[ I_g = I_0(e'_1 \otimes e'_1 + e'_2 \otimes e'_2) + I_1 e'_3 \otimes e'_3 . \]

Angular velocity vector of gyroscope:

\[ \omega_g = \dot{\theta}e'_1 + \dot{\phi} \sin(\theta)e'_2 + (\dot{\psi} + \dot{\phi} \cos(\theta))e'_3 , \]

In frame \( F' \) we can use Euler’s equations to represent this balance. In the fixed frame \( F \), those equations read as:

\[
M_1 = I_0 \frac{d}{dt} \left[ -\dot{\phi} \sin(\phi) \sin(\theta) \cos(\theta) + \dot{\theta} \cos(\phi) \right] + I_1 \frac{d}{dt} \left[ \sin(\theta) \sin(\phi)(\dot{\phi} \cos(\theta) + \dot{\psi}) \right],
\]

\[
M_2 = I_0 \frac{d}{dt} \left[ \dot{\phi} \cos(\phi) \sin(\theta) \cos(\theta) + \dot{\theta} \sin(\phi) \right] - I_1 \frac{d}{dt} \left[ \sin(\theta) \cos(\phi)(\dot{\phi} \cos(\theta) + \dot{\psi}) \right],
\]

\[
M_3 = I_0 \frac{d}{dt} (\dot{\phi} \sin^2(\theta)) + I_1 \frac{d}{dt} \left[ \cos(\theta)(\dot{\psi} + \dot{\phi} \cos(\theta)) \right].
\]
Linear momentum balance for the mass

Here, \( F \) represents the force supplied to the mass by the gyroscope, that we should determine.

Nonlinear restoring force of the rods

\[
\mathbf{h}[\mathbf{u}(t)] = \sum_{i=1}^{3} \left[ (|\mathbf{u}(t) - \mathbf{L}a^{(i)}| - L) \frac{\mathbf{u}(t) - \mathbf{L}a^{(i)}}{|\mathbf{u}(t) - \mathbf{L}a^{(i)}|} + (|\mathbf{u}(t) + \mathbf{L}a^{(i)}| - L) \frac{\mathbf{u}(t) + \mathbf{L}a^{(i)}}{|\mathbf{u}(t) + \mathbf{L}a^{(i)}|} \right],
\]

In going forward we use the following normalisations:

\[
\tilde{h} = \frac{h}{L}, \quad \tilde{F} = \frac{F}{cL}, \quad \tilde{M}_e = \frac{M_e}{cLl}, \quad \tilde{I}_j = \frac{I_j}{mlL} \quad (j = 0, 1),
\]

and

\[
\tilde{u} = \frac{u}{L}, \quad \tilde{l} = \frac{l}{l}, \quad \tilde{t} = \sqrt{\frac{c}{m\gamma}} t, \quad \gamma = 1 + \frac{I_0}{ml^2}, \quad \delta = \frac{l}{L}.
\]

to obtain all quantities in the dimensionless form.
Assumptions on the motion of the system

Position vector of gyroscope tip:

\[ l(t) = \delta \sin(\theta(t)) \sin(\phi(t)) e_1 - \delta \sin(\theta(t)) \cos(\phi(t)) e_2 + \delta \cos(\theta(t)) e_3, \]

We assume

I. The connection of the gyroscope with the mass is such that

\[ u(t) = l(t) - l(0) = \delta \sin(\theta(t)) \sin(\phi(t)) e_1 - \delta \sin(\theta(t)) \cos(\phi(t)) e_2 + \delta (\cos(\theta(t)) - 1) e_3, \]

II. The nutation angle of the gyroscope and its derivatives satisfy

\[ \left| \frac{d^j \theta(t)}{dt^j} \right| \leq \text{Const } \varepsilon, \quad j = 0, 1, 2, \]
Governing equation for the mass

Application of assumption II (nutation angle is small) we find that

\[ F_3 = O(\varepsilon^2) , \quad u_3 = O(\varepsilon^2) , \quad M_3 = O(\varepsilon^2) \]

and from the linear momentum balance we have

\[ F = -Ku(t) - \gamma^{-1} \ddot{u}(t) + O(\varepsilon^2) , \quad K = 2 \sum_{j=1}^{3} a^{(j)} \otimes a^{(j)} = 3 I_2 , \]

now \( F = (F_1, F_2)^T \) and \( u = (u_1, u_2)^T \). On the other hand, from assumptions I and II with the angular momentum balance give

\[
F_1 = \frac{I_0}{\gamma \delta} \ddot{u}_1 + \frac{I_1}{\gamma \delta} [ (\ddot{\psi} + \dot{\phi})u_2 + (\dot{\psi} + \dot{\phi})\dot{u}_2 ] + O(\varepsilon^3) ,
\]

\[
F_2 = \frac{I_0}{\gamma \delta} \ddot{u}_2 - \frac{I_1}{\gamma \delta} [ (\ddot{\psi} + \dot{\phi})u_1 + (\dot{\psi} + \dot{\phi})\dot{u}_1 ] + O(\varepsilon^3) ,
\]

We have to leading order the sum of the precession and nutation rate is constant. We define

\[ \alpha = \frac{I_1}{(\delta + I_0)} \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \Omega = \dot{\psi}(0) + \dot{\phi}(0) . \]

where \( \Omega \) is the gyricity of the gyroscope. Combining the above we obtain

**Governing equation for the mass:**

\[ \ddot{u}(t) + \alpha \Omega R \dot{u}(t) + 3u(t) = 0 \]
Eigenmode analysis of the system

Equation of motion of the mass is:

\[ \ddot{u}(t) + \alpha \Omega R \dot{u}(t) + 3u(t) = 0 \quad \text{with} \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Introduce the time harmonic solution:

\[ u(t) = A e^{i\omega t}, \]

The non-trivial eigenfrequencies and eigenvectors are then:

\[ \omega_{\pm} = \frac{1}{2} \left[ \pm \alpha \Omega + \sqrt{(\alpha \Omega)^2 + 12} \right], \quad A(\omega) = \begin{pmatrix} 1 \\ i \frac{3 - \omega^2}{\alpha \Omega \omega} \end{pmatrix}. \]

**Example:** We set \( \alpha = 0.25 \) and observe dependency of the eigenfrequencies on \( \Omega \) and some of the modes.
Predicting the motion of the system

\[ u(t) = c_1 A(\omega_+) e^{i\omega t} + c_2 \overline{A(\omega_+)} e^{-i\omega t} + c_3 A(\omega_-) e^{i\omega t} + c_4 \overline{A(\omega_-)} e^{-i\omega t}, \]

**Example:** We take \( \Omega = 6, \alpha = 0.25 \) and determine the behaviour of the system after release from some initial configuration.

\[ u(0) = \begin{pmatrix} 0 \\ -0.05 \end{pmatrix}, \quad \dot{u}(0) = \begin{pmatrix} 0.05 \\ 0 \end{pmatrix}. \]
Validation against independent FE simulations

Initial conditions:
\[ u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dot{u}(0) = \begin{pmatrix} 0 \\ -0.040 \end{pmatrix} \]
Gyroelastic lattice. Equation of motion

\[-\frac{m\omega^2}{c}u^{(n)} = a^{(1)} \cdot (u^{(n+e_1)} - u^{(n)})a^{(1)} + (-a^{(1)}) \cdot (u^{(n-e_1)} - u^{(n)})(-a^{(1)}) + a^{(2)} \cdot (u^{(n-e_1+e_2)} - u^{(n)})a^{(2)} + (-a^{(2)}) \cdot (u^{(n+e_1-e_2)} - u^{(n)})(-a^{(2)}) + a^{(3)} \cdot (u^{(n-e_2)} - u^{(n)})a^{(3)} + (-a^{(3)}) \cdot (u^{(n+e_2)} - u^{(n)})(-a^{(3)}) + \frac{\alpha i\omega^2}{c}Ru^{(n)}\]

where \( R \) is the rotation matrix

\[R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\]

and \( \alpha \) is the spinners constant
Dispersion equation

\[
\det \left[ C(k) - \omega^2 (M - \Sigma) \right] = 0
\]

where \( M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \) and \( \Sigma = \begin{pmatrix} 0 & -i\alpha \\ i\alpha & 0 \end{pmatrix} \)

Stiffness matrix

\[
C(k) = c \begin{pmatrix}
3 - 2 \cos k_1 l - \frac{(\cos \zeta + \cos \xi)}{2} & \frac{\sqrt{3}(\cos \xi - \cos \zeta)}{2} \\
\frac{\sqrt{3}(\cos \xi - \cos \zeta)}{2} & 3 - \frac{3(\cos \zeta + \cos \xi)}{2}
\end{pmatrix}
\]

where \( \zeta = \frac{k_1 l}{2} + \frac{\sqrt{3}}{2} k_2 l \) and \( \xi = \frac{k_1 l}{2} - \frac{\sqrt{3}}{2} k_2 l \)
Explicitly

\[ \omega^4(m^2 - \alpha^2) - \omega^2 m \text{tr}C + \text{det}C = 0 \]

\[
\begin{align*}
\omega_1(k) &= \sqrt{\frac{\text{tr}(C) - \sqrt{\text{tr}^2(C) - 4(1 - (\alpha/m)^2) \text{det}(C)}}{2(1 - (\alpha/m)^2)}} \\
\omega_2(k) &= \sqrt{\frac{\text{tr}(C) + \sqrt{\text{tr}^2(C) - 4(1 - (\alpha/m)^2) \text{det}(C)}}{2(1 - (\alpha/m)^2)}}
\end{align*}
\]

Three Regimes.

- \( \alpha^2 < m^2 \) \textit{subcritical}: two dispersion surfaces.
- \( \alpha^2 > m^2 \) \textit{supercritical}: one dispersion surface.
- \( \alpha^2 = m^2 \) \textit{critical}: one dispersion surface

\[
\omega = (m^{-1} \text{det}C/\text{tr}C)^{1/2}
\]